PROPAGATION OF A SHEAR CRACK IN A RANDOMLY HETEROGENEOUS BODY

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Propagation of a crack in a randomly heterogeneous body exposed to longitudinal shear is considered (in a Born approximation). It is proved that the stress means at the crack tip have singularities on the order of $(r)^{-1/2}$. The effective coefficient of stress intensity is introduced. It is known that the propagation of a crack in a homogeneous body is of a local nature, i.e., energy consumption in the growth of the crack is completely determined by the coefficient of stress intensity, which is a local characteristic. The equivalence of the force and energy approaches is mathematically expressed by the Irwin equation [1]. An analog of the Irwin equation is obtained for the case of a randomly heterogeneous body.

Let us consider an elastic heterogeneous body containing a crack situated along the x axis and undergoing longitudinal shear. In this case $u=v=0$, $w=w(x, y)\neq 0$, $\sigma_x=\sigma_y=\tau_{xy}=\sigma_z=0$, $\tau_{xz}\neq 0$, and $\tau_{zy}\neq 0$. The equilibrium equation and Hooke's law are written in the form

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}=0;\ \tau_{xz}=\mu\frac{\partial w}{\partial x};\ \tau_{yz}=\mu\frac{\partial w}{\partial y} \tag{1}$$

under the boundary condition $\tau_{xz}\cos nx+\tau_{yz}\cos ny=f$.

The Griffith crack criterion was given in [2, 3] in integral form and, in the case of longitudinal shear, is written in the form

$$\frac{\delta W}{\delta l}=\frac{1}{2}R\int\limits_0^{2\pi}\left(W_0\cos\theta-\tau_{xz}\frac{\partial w}{\partial x}\cos\theta-\tau_{yz}\frac{\partial w}{\partial x}\sin\theta\right)d\theta=G;$$

W_0 is the specific elastic potential and R is the radius of a circle about the crack tip.

Suppose the shear modulus μ is a random function of the x and y coordinates and is independent of z. We may represent the shear modulus in the form $\mu=\langle\mu\rangle+\mu'$ and assume that fluctuations of μ' are small in terms of the standard deviation in comparison with $\langle\mu\rangle$. If we set $u=\langle w\rangle$ and $v=w-u$, after substituting the expressions for μ in Eq. (1), we obtain a statistically nonlinear problem that can be linearized if the solution is represented in the form of a series in powers of some parameter $\varkappa$ [4].

If we limit ourselves to two terms of the decomposition of the displacements w, setting $\varkappa=1$, we may prove that the mean value of the elastic energy increment per unit length of the crack has the form

$$\frac{\delta\langle W\rangle}{\delta l}=R\int\limits_0^{2\pi}\Big[\frac{1}{2}\Big(-\langle\mu\rangle\varepsilon_{xz}^2+\langle\mu\rangle\Big\langle\frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\Big\rangle-\langle\mu\rangle\Big\langle\frac{\partial v}{\partial x}\frac{\partial v}{\partial x}\Big\rangle-2\Big\langle\mu'\frac{\partial v}{\partial x}\Big\rangle\varepsilon_{xz}+\langle\mu\rangle\varepsilon_{yz}^2$$
$$+2\Big\langle\mu'\frac{\partial v}{\partial y}\Big\rangle\varepsilon_{yz}\Big)\cos\theta-\Big(\langle\mu\rangle\varepsilon_{xz}\varepsilon_{yz}+\langle\mu\rangle\Big\langle\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}\Big\rangle+\Big\langle\mu'\frac{\partial v}{\partial y}\Big\rangle\varepsilon_{xz}+\Big\langle\mu'\frac{\partial v}{\partial x}\Big\rangle\varepsilon_{yz}\Big)\sin\theta\Big]d\theta, \tag{2}$$
$$\varepsilon_{xz}=\frac{1}{2}\frac{\partial u}{\partial x},\ \varepsilon_{yz}=\frac{1}{2}\frac{\partial u}{\partial y}.$$

It is necessary to solve a boundary-value problem for the mean displacements u in order to calculate all the variables occurring in Eq. (2):

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{3}$$
$$\langle \mu \rangle (\varepsilon_{xz} \cos nx + \varepsilon_{yz} \cos ny) = f$$

and a boundary-value problem for the motion fluctuations

$$\langle\mu\rangle\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = -\frac{\partial}{\partial x}(\mu'\varepsilon_{xz}) - \frac{\partial}{\partial y}(\mu'\varepsilon_{yz}), \tag{4}$$
$$\langle\mu\rangle\left(\frac{\partial v}{\partial x}\cos nx + \frac{\partial v}{\partial y}\cos ny\right) = -\mu'(\varepsilon_{xz}\cos n x + \varepsilon_{yz}\cos ny).$$

The solution of the problem (3) is known and the solution of Eq. (4) can be represented in terms of the Green's function $G(x, y, x_0, y_0)$ of a problem from homogeneous elasticity theory (3) [4]

$$v(x, y) = \frac{1}{\langle\mu\rangle}\int_S G(x, y, x_0, y_0)\left\{\frac{\partial}{\partial x_0}[\mu'(x_0, y_0)\varepsilon_{xz}(x_0, y_0)]\right.$$
$$\left.+\frac{\partial}{\partial y_0}[\mu'(x_0, y_0)\varepsilon_{yz}(x_0, y_0)]\right\}dx_0dy_0 - \frac{1}{\langle\mu\rangle}\int_L G(x, y, x_0, y_0)$$
$$\times\{\mu'(x_0, y_0)[\varepsilon_{xz}(x_0, y_0)\cos nx + \varepsilon_{yz}(x_0, y_0)\cos ny]\}\,dL.$$

We determine the stress means from the equations

$$\langle\tau_{xz}\rangle = \langle\mu\rangle\varepsilon_{xz} + \left\langle\mu'\frac{\partial v}{\partial x}\right\rangle; \quad \langle\tau_{yz}\rangle\langle\mu\rangle\varepsilon_{yz} + \left\langle\mu'\frac{dv}{\partial y}\right\rangle; \tag{5}$$
$$\langle\tau_{xz}\rangle = \langle\mu\rangle\varepsilon_{xz} + \frac{1}{\langle\mu\rangle}\int_S f_1\frac{\partial G}{\partial x}dx_0dy_0 - \frac{1}{\langle\mu\rangle}\int_L f_2\frac{\partial G}{\partial x}dL;$$
$$\langle\tau_{yz}\rangle = \langle\mu\rangle\varepsilon_{yz} + \frac{1}{\langle\mu\rangle}\int_S f_1\frac{\partial G}{\partial y}dx_0dy_0 - \frac{1}{\langle\mu\rangle}\int_L f_2\frac{\partial G}{\partial y}dL;$$
$$f_1 = \left\langle\mu'(x, y)\frac{\partial\mu'(x_0, y_0)}{\partial x_0}\right\rangle\varepsilon_{xz}(x_0, y_0) + \left\langle\mu'(x, y)\frac{\partial\mu'(x_0, y_0)}{\partial y_0}\right\rangle\varepsilon_{yz}(x_0, y_0);$$
$$f_2 = -\langle\mu'(x, y)\mu'(x_0, y_0)\rangle[\varepsilon_{xz}(x_0, y_0)\cos nx + \varepsilon_{yz}(x_0, y_0)\cos ny],$$

where S is the region outside the section and L is the section line.

Suppose we have an infinite body with a semiinfinite crack loaded with a concentrated force P at a distance h units from its tip (Fig. 1). It is required to determine the Green's function of the harmonic problem, such that its derivative $\partial G(x, y, x_0, y_0)/\partial n = 0$ along the section line. The Green's function of two variables $G(x, y, x_0, y_0)$ has the form [5]

$$G(x, y, x_0, y_0) = \frac{1}{2\pi}\left[\ln\frac{1}{\rho} + \varphi(x, y, x_0, y_0)\right],$$
$$\rho = \{(x - x_0)^2 + (y - y_0)^2\}^{1/2}.$$

The derivatives of the Green's function occurring in Eq. (5) have the form

$$\frac{\partial G}{\partial x} = \frac{1}{2\pi}\left[-\frac{x - x_0}{(x - x_0)^2 - (y - y_0)^2} + \frac{\partial\varphi(x, y, x_0, y_0)}{\partial x}\right];$$
$$\frac{\partial G}{\partial y} = \frac{1}{2\pi}\left[-\frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2} + \frac{\partial\varphi(x, y, x_0, y_0)}{\partial y}\right].$$

The harmonic function $\varphi(x, y, x_0, y_0)$ is the real part of some function f(z) analytic within the region occupied by the body,

$$\varphi = \mathrm{Re}\{f(z)\}; \quad \frac{\partial\varphi}{\partial x} = \mathrm{Re}\{f'(z)\}; \quad \frac{\partial\varphi}{\partial y} = -\mathrm{Im}\{f'(z)\}$$

and satisfies the boundary condition

$$\mathrm{Im}\{f'(z)\}\Big|_{\substack{y=0\\ x\in[0,\infty]}} = \frac{\partial}{\partial y}\left(\ln\frac{1}{\rho}\right)\Big|_{\substack{y=0\\ x\in[0,\infty]}}.$$

We use the Keldysh–Sedov formula [6] to calculate the function f'(z) by means of the equation

$$f'(z) = \frac{y_0}{2\pi^2\sqrt{z}}\int_0^\infty\frac{\sqrt{p}\,dp}{(p - z)\left[(p - x_0)^2 + y_0^2\right]}.$$

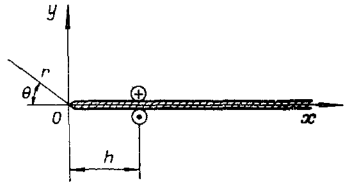

Fig. 1

The equations for ε_{xz} and ε_{yz} corresponding to the homogeneous problem can be obtained from [7].

As a result of the calculations, we find that the stress means in the neighborhood of the crack tip have the form

$$\langle\tau_{xz}\rangle = \frac{P\sin\theta/2}{\sqrt{2r}}F;\ \langle\tau_{yz}\rangle = \frac{P\cos\theta/2}{\sqrt{2r}}F; \tag{6}$$

$$F = \left[\frac{1}{\pi\sqrt{2h}} + \frac{1}{\pi\langle\mu\rangle^2}\int_s \frac{\sqrt{\sqrt{x_0^2+y_0^2}+x_0}}{\sqrt{x_0^2+y_0^2}}\left[\left\langle\mu'(0,0)\frac{\partial\mu'(x_0,y_0)}{\partial x_0}\right\rangle\right.\right.$$
$$\times\,\varepsilon_{xz}(x_0,y_0) + \left.\left\langle\mu'(0,0)\frac{\partial\mu'(x_0,y_0)}{\partial y_0}\right\rangle\varepsilon_{yz}(x_0,y_0)\right]dx_0dy_0$$
$$-\frac{\sqrt{2}}{\pi\langle\mu\rangle}\int_0^\infty\frac{\delta(x_0-h)\langle\mu'(0,0)\,\mu'(x_0,0)\rangle}{\sqrt{x_0}}dx_0;\ k_3 = \frac{P}{\pi\sqrt{2h}}.$$

It is clear from Eqs. (6) that the stresses have a singularity on the order of $(r)^{-1/2}$ in a neighborhood of the crack tip.

Let us introduce the effective coefficient of stress intensity k'_3 and write Eqs. (6) in the form

$$\langle\tau_{xz}\rangle = \frac{\sin(\theta/2)}{\sqrt{2r}}k_3';\ \langle\tau_{yz}\rangle = \frac{\cos(\theta/2)}{\sqrt{2r}}k_3'; \tag{7}$$
$$k_3' = PF = k_3 + k_3^*.$$

The first term in the right side of the second formula in Eqs. (7) is the known value of the stress-intensity coefficient for a homogeneous body, while the second term represents the addition due to the random heterogeneity of the body.

We now turn to Eq. (2), assume that the radius R is small, and use equations (7), obtaining

$$\varepsilon_{xz}^2 = \frac{\sin^2(\theta/2)}{2\langle\mu\rangle^2R}k_3^2;\ \varepsilon_{xz}\left\langle\mu'\frac{\partial v}{\partial x}\right\rangle = \frac{\sin^2(\theta/2)}{2\langle\mu\rangle R}k_3k_3^*;$$
$$\left\langle\frac{\partial v}{\partial x}\frac{\partial v}{\partial x}\right\rangle = \frac{\sin^2(\theta/2)}{2\langle\mu\rangle^2R}\left(k_3^*k_3^*\right);$$
$$\left\langle\frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right\rangle = \frac{\cos^2(\theta/2)}{2\langle\mu\rangle^2R}\left(k_3^*k_3^*\right),$$

where .

$$k_3^*k_3^* = \int_S\int_S\frac{\sqrt{\sqrt{x_0^2+y_0^2}+x_0}\sqrt{\sqrt{x_1^2+y_1^2}+x_1}}{\sqrt{x_0^2+y_0^2}\sqrt{x_1^2+y_1^2}}$$
$$\times\left[\left\langle\frac{\partial\mu'(x_0,y_0)}{\partial x_0}\frac{\partial\mu'(x_1,y_1)}{\partial x_1}\right\rangle\varepsilon_{xz}(x_0,y_0)\,\varepsilon_{xz}(x_1,y_1) + \left\langle\frac{\partial\mu'(x_1,y_1)}{\partial y_1}\right.\right.$$
$$\times\left.\frac{\partial\mu'(x_0,y_0)}{\partial y_0}\right\rangle\varepsilon_{yz}(x_0,y_0)\,\varepsilon_{yz}(x_1,y_1) + \left\langle\frac{\partial\mu'(x_0,y_0)}{\partial y_0}\frac{\partial\mu'(x_1,y_1)}{\partial x_1}\right\rangle\varepsilon_{yz}(x_0,y_0)$$
$$\times\left.\varepsilon_{xz}(x_1,y_1) + \left\langle\frac{\partial\mu'(x_1,y_1)}{\partial y_1}\frac{\partial\mu'(x_0,y_0)}{\partial x_0}\right\rangle\varepsilon_{yz}(x_1,y_1)\,\varepsilon_{xz}(x_0,y_0)\right]$$
$$\times\,dx_0dy_0dx_1dy_1 - \frac{2\sqrt{2}}{\sqrt{h}}\int_S\left\{\left\langle\frac{\partial\mu'(x_0,y_0)}{\partial x_0}\mu'(h,0)\right\rangle\varepsilon_{xz}(x_0,y_0)\right.$$
$$\left.+\left\langle\frac{\partial\mu'(x_0,y_0)}{\partial y_0}\mu'(h,0)\right\rangle\varepsilon_{yz}(x_0,y_0)\right\}dx_0dy_0. \tag{8}$$

Following some algebra, Eq. (2) takes the form

$$\frac{\delta \langle W \rangle}{\delta l} = \frac{\pi}{2\langle \mu \rangle}\{k_3^2 + 2k_3 k_3^* + \langle k_3^* k_3^* \rangle\} = G. \tag{9}$$

The expression within braces in Eqs. (8) and (9) differs from the complete square of k'_3 by the presence of a correlation between fluctuations of the elastic constants at different points of the body.

Thus, the Griffith–Irwin crack criterion is described by Eq. (9) for a longitudinal crack in a randomly heterogeneous body, i.e., the crack begins to grow at the point when the function of the local characteristics occurring within the braces reaches a value G.

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THERMOMECHANICAL BEHAVIOR OF A RECTANGULAR VISCOELASTIC PRISM EXPOSED TO REPEATED STRETCHING AND CONTRACTION

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Vibrational heat production is an important problem in studying the efficiency of viscoelastic elements of structures experiencing cyclic loads. The design of heat regimes constitutes one of the fundamental problems in the construction of such types of vibrational-proof systems as laminated rods, plates, and shells [1] and fiberglass and rubber–metal products, in particular, shock absorbers [2, 3]. Calculation of the critical parameters beyond which a rapid growth in temperature occurs (the phenomenon of thermal explosion), which leads to partial or complete loss of the supporting power of the product as a result of softening of the material, is of particular interest. A variational method has been used [4] to calculate heat production in a two-dimensional shock absorber. The boundary conditions are satisfied on the basis of the St. Venant principle. In the current work, the stress–strain state, self-heating temperature field, and thermal instability of a long rectangular prism being periodically loaded (plane deformation) are investigated.

§ 1. The fundamental thermoelastic equations are presented in [5]. We may obtain the fundamental thermoviscoelastic equations when ν = const by replacing the shear modulus μ by an operator μ^*. We will find the solution of these equations for a plate $|\xi| \leq 2L$, $|\eta| \leq 2H$ under the boundary conditions

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